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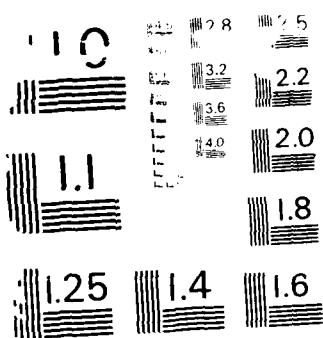
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Final Report to the U.S. Air Force Office of Scientific Research

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"Statistical Description of Stochastic Dynamics"

Principle Investigator Alexander B. Rechester

The main result of our research is the establishment of a general relationship for fluctuation of the spectral density of the chaotic motion which is similar to the Einstein fluctuation formula in statistical mechanics. We have introduced a Gibbs-type partition of the chaotic motion. The distribution function of the spectral density defined on such partition is Gaussian. The variance of this distribution is the Fourier transform of the correlation function. We demonstrate this by direct numerical computations for the simple models of chaos. These results are the consequence of translational invariance and should be valid for the general case of chaotic motion described by differential equations. More details of this work could be found in the enclosed paper "Gibbs Type Partition in the Chaotic Dynamics" by L. Bromberg and A. B. Rechester, Phys. Rev. A 37, 1708 (1988).

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## Gibbs-type partition in chaotic dynamics

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A Gibbs-type partition of the chaotic motion is introduced. The distribution function of the spectral density defined on such partition is Gaussian. The variance of this distribution is the Fourier transform of the correlation function. We demonstrate this by direct numerical computations for the simple models of chaos. These results are the consequence of translational invariance and should be valid for the general case of chaotic motion described by differential equations.

The purpose of this paper is to establish a general relationship for fluctuation of the spectral density of the chaotic motion, which is similar to the Einstein fluctuation formula in statistical mechanics.<sup>1</sup>

We begin with the simplest model of chaotic motion given by one-dimensional quadratic mapping<sup>2</sup>

$$x_{t+1} = \alpha x_t (1 - x_t). \quad (1)$$

Here,  $t = 0, 1, 2, \dots$  can be thought as a discrete time. The iterative sequence produced by Eq. (1),  $\{x_0, x_1, x_2, \dots, x_T\}$ , will be called an orbit;  $T$  is the time length of the orbit and  $x_0$  its initial condition. It is easy to see that for the values of parameter  $0 < \alpha < 4$ , the orbits are bounded,  $0 < x_t < 1$ . The chaotic orbits are characterized by a positive Liapunov number  $\lambda > 0$ .<sup>2</sup>

$$\lambda = \lim_{T \rightarrow \infty} \left( \frac{1}{T} \sum_{t=0}^{T-1} \ln \left| \frac{dx_{t+1}}{dx_t} \right| \right). \quad (2)$$

On the other hand, in the case of regular (periodic) orbits  $\lambda \leq 0$ . Another important property of chaotic orbits is their power spectrum. The Fourier transform for a discrete variable  $x_t$  can be written in the form

$$a_n = \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} x_t e^{-i\omega_n t}, \quad (3)$$

$$x_t = \frac{1}{\sqrt{T}} \sum_{n=0}^{T-1} a_n e^{i\omega_n t}. \quad (4)$$

Here,  $\omega_n = 2\pi n / T$ ,  $a_{-n} = a_n^*$ . Consider the limit  $T \rightarrow \infty$ . In the case of chaotic orbits, the  $\omega$  spectrum becomes continuous and we can substitute

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=0}^{T-1} \rightarrow \frac{1}{2\pi} \int_0^{2\pi} d\omega. \quad (5)$$

In the case of periodic orbits,  $a(\omega)$  is discrete (in the limit  $T \rightarrow \infty$ ), that is, a sum of delta functions. Often, chaotic motion also exhibits delta-functional peaks in the spectrum. For example, usually,  $a_0 = \langle x_t \rangle \sqrt{T} \rightarrow \delta(\omega)$  when  $T \rightarrow \infty$ . Such delta-functional peaks could also happen for a finite value of  $\omega$ .<sup>3</sup>

Introduce now a correlation function:

$$C(\tau) = \langle x_{t+\tau} x_t \rangle - \langle x_t \rangle^2. \quad (6)$$

The time averaging is defined by the equation

$$\langle A(x_t) \rangle = \lim_{T \rightarrow \infty} \left( \frac{1}{T} \sum_{t=0}^{T-1} A(x_t) \right). \quad (7)$$

Here  $A(x)$  is some function of  $x$ . The correlation function  $C(\tau, T)$  averaged over the finite interval  $T$  is simply related to power spectrum. Using Eqs. (4), (6), and (7) we can get

$$C(\tau, T) = \frac{1}{T} \sum_{n=1}^{T-1} |a(\omega_n)|^2 e^{i\omega_n \tau}. \quad (8)$$

Notice that the term  $a_0 = \langle x_t \rangle \sqrt{T}$  is absent here due to the subtraction in Eq. (6). We have done detailed numerical studies of the correlation function  $C(\tau, T)$  and the power spectrum  $|a(\omega_n)|^2$  for simple models of chaotic motion. The typical behavior of  $C(\tau, T)$  is plotted in Fig. 1 for a quadratic mapping. The value of the parameter  $\alpha = 3.94$  corresponds to a chaotic attractor with a positive Liapunov number  $\lambda = 0.54$ . In Fig. 1,  $T = 2000$ . First of all, there exists numerically a well-defined limit  $C(\tau, T \rightarrow \infty) = C(\tau)$ . Long-time behavior of  $C(\tau \rightarrow \infty)$  in many cases can be approximated as an oscillation and an exponential decay with characteristic time  $\tau_c$ .

The plot of a power spectrum  $|a(\omega_n)|^2$  for the same value of parameters is given in Fig. 2. Due to the discreteness of our model and the symmetry condition  $a(-\omega_n) = a^*(\omega_n)$  we have plotted it only in the interval

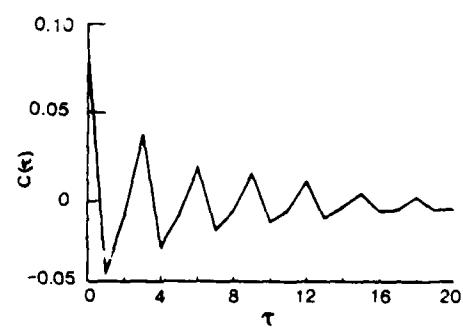


FIG. 1. Correlation function  $C(\tau)$  computed for quadratic mapping, Eq. (1). The values of parameters are  $\alpha = 3.94$ ,  $T = 2000$ , and  $\lambda = 0.54$ .

$0 < \omega < \pi$ . It is obvious from this plot that  $a(\omega_n)$  is a fluctuating quantity. A small change in the initial condition of the orbit  $x_0$  or in the value of two adjacent frequencies  $2\pi n/T, 2\pi(n+1)/T$  will usually result in an abrupt change in the value of  $a(\omega_n)$ . Most importantly, there exists no limit  $a(\omega_n, T \rightarrow \infty)$ . Thus, the left-hand side of Eq. (8) is well-defined in the limit  $T \rightarrow \infty$  but each term in the sum on the right-hand side is not. From the mathematical point of view we have to do some resummation in the right-hand side of Eq. (8) in order to have an average quantity  $\langle |a(\omega_n)|^2 \rangle$  which converges in the limit  $T \rightarrow \infty$  to the Fourier transform of the correlation function. From the physics point of view this means choosing an ensemble over which such averaging can be done. We will introduce now such an ensemble. Let us use as a new time of averaging  $MT$  instead of  $T$  in Eq. (8);

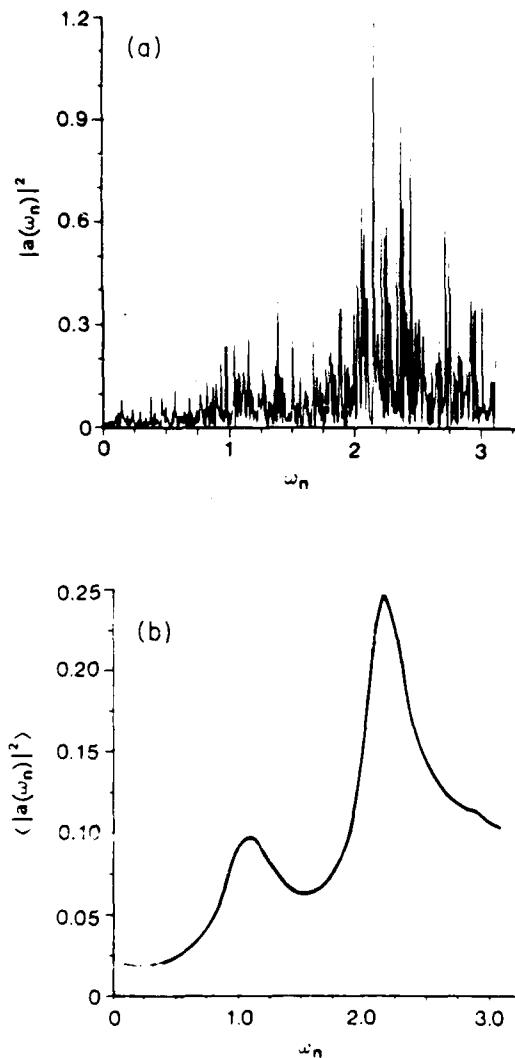


FIG. 2. (a) Power spectrum  $|a(\omega_n)|^2$  computed for the same case:  $\alpha = 3.94$ ,  $T = 128$ . (b) Average power spectrum  $\langle |a(\omega_n)|^2 \rangle$  computed for the same case  $\alpha = 3.94$ ,  $T = 128$ , and  $M = 10^5$ .

here  $M \gg 1$  is a very large integer. We also assume that  $T \gg \tau_c$ . Now we split the time averaging in Eq. (7) into two parts

$$\langle A \rangle_{MT} = \frac{1}{M} \sum_{m=1}^M \frac{1}{T} \sum_{t=T(m-1)}^{Tm} A(x_t) = \langle \langle A \rangle_T \rangle_G. \quad (9)$$

Here, the brackets with subscript  $T$  denote usual time averaging, while brackets with the subscript  $G$  denote averaging over the ensemble of  $M$  test orbits. We call this a Gibbs-type ensemble because in the case of a one-dimensional lattice when our discrete time is changed to a discrete spacing, it is equivalent to a Gibbs partition of statistical mechanics.<sup>1</sup> Then, we can rewrite Eq. (8) in the form

$$C(\tau, MT) = \frac{1}{T} \sum_{n=1}^{T-1} \langle |a(\omega_n)|^2 \rangle_G e^{i\omega_n \tau}. \quad (10)$$

We have computed numerically the value of  $\langle |a(\omega_n)|^2 \rangle$  (the subscript  $G$  will be omitted from now on) as a function of  $\omega_n$  for the example given in the previous Figs. 1 and 2(a). It is plotted in Fig. 2(b) for  $\alpha = 3.94$ ,  $T = 128$ , and  $M = 10^5$ . The average power spectrum converges numerically to a well-defined continuous function of  $\omega$  in the limit  $M \rightarrow \infty$ ,  $T \rightarrow \infty$ . This function is a statistical property of the orbit and is independent of the initial condition  $x_0$ . We should mention here that in general several attractors could be present, then depending on the value of initial condition  $x_0$  the orbit will evolve on the different attractors and could have different values of  $\langle |a(\omega_n)|^2 \rangle$ . Consider now the limit  $M \rightarrow \infty$ ,  $T \rightarrow \infty$ . Then we can write Eq. (8) using Eq. (5) in the form

$$C(\tau) = \frac{1}{2\pi} \int_0^{2\pi} G(\omega) e^{i\omega \tau} d\omega, \quad \tau > 0 \quad (11)$$

$$G(\omega) = \langle |a(\omega)|^2 \rangle = \sum_{\tau=0}^{\infty} C(\tau) e^{-i\omega \tau}. \quad (12)$$

The singularity of  $G(\omega)$  in the complex plane  $z = e^{i\omega}$  which is closest to the unit circle  $|z| = 1$  determines the long-time behavior of  $C(\tau \rightarrow \infty)$ . In the case of a pole  $z_0 = \exp(-1/\tau_c + i\omega_0)$ ,  $|z_0| < 1$ , we will have

$$C(\tau \rightarrow \infty) \sim \cos(\omega_0 \tau + \varphi_0) e^{-\tau/\tau_c}. \quad (13)$$

We will turn now to the numerical computations of the distribution function  $P(a_n)$  of a Fourier coefficient  $a_n$  with a given frequency  $\omega_n = 2\pi n/T$ . We compute this distribution based on Gibbs-type ensemble of  $M$  test orbits according to

$$P(\xi_n, \zeta_n) = \frac{1}{M} \frac{\Delta M}{\Delta \xi_n \Delta \zeta_n}, \quad (14)$$

$$a_n = \xi_n + i\zeta_n.$$

Here,  $\Delta M$  is a number of test orbits in the interval  $\Delta \xi_n$  and  $\Delta \zeta_n$ . We have done extensive numerical studies of this distribution for many different chaotic attractors described by Eq. (1) and for other mappings.<sup>4</sup> The typical results of these computations are presented in Figs. 3 and 4. Figure 3 is a contour plot of the distribution function  $P(a_n)$ . These computations were done for a quadratic

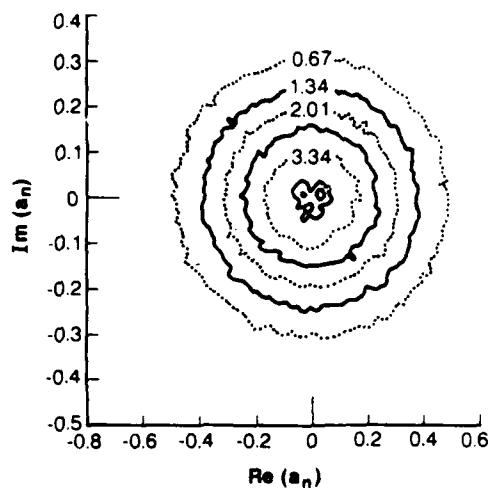


FIG. 3. Contour plots of the distribution function  $P(a_n)$  computed for quadratic mapping:  $\alpha=3.94$ ,  $T=128$ ,  $\omega=0.785$ .

mapping Eq. (9) with the values of parameters:  $\alpha=3.94$ ,  $T=128$ ,  $\omega=0.785$ ,  $M=10^6$ . It is clear from Fig. 3 that  $P(a_n)$  is axisymmetric. The dependence of  $P(|a_n|)$  has been computed in Fig. 4. The values of the parameters for this computation were the same as for Fig. 3, except that  $\omega=1.23$ . The results of these computations can be parametrized in the form

$$P(|a_n|) = \frac{2|a_n|}{\langle |a_n|^2 \rangle} \exp \left( -\frac{|a_n|^2}{\langle |a_n|^2 \rangle} \right). \quad (15)$$

We would like now to show that Eq. (15) is a consequence of the translational invariance of our distribution. Let us make a transformation  $x'_i = x_i + \tau$ , where  $\tau$  is an arbitrary integer. In Fourier space this corresponds to transformation

$$a'_n = |a_n| \exp[i(\varphi_n + \omega_n \tau)]. \quad (16)$$

Distribution function  $P(a_n)$  should be invariant under such transformation. This implies that it is independent of phase  $\varphi_n$ . The Fourier coefficient for the whole interval  $MT$  can be expressed through our ensemble as

$$a(\omega_n, MT) = \frac{1}{\sqrt{M}} \sum_{m=1}^M |a_n^m| e^{i\varphi_n^m}. \quad (17)$$

Phases  $\varphi_n^m$  in the sum (17) are uniformly distributed. Then, based on the central-limit theorem we can claim

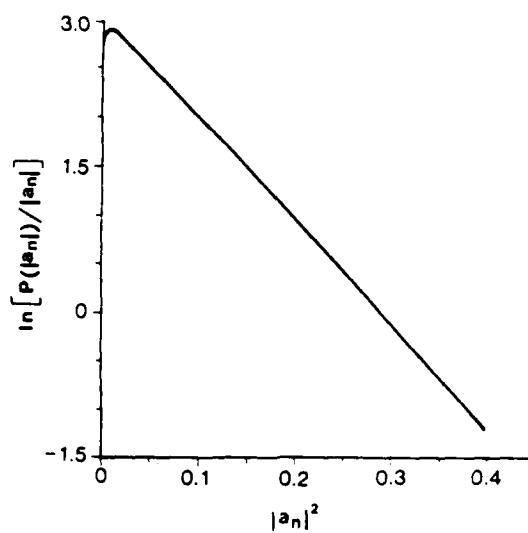


FIG. 4. Distribution function of the spectral density computed numerically for the case of quadratic mapping  $\alpha=3.94$ ,  $T=128$ ,  $\omega=1.23$ , and  $M=10^6$ . Its functional form is given by Eq. (15).

that the real and imaginary part of  $a_n$  have a Gaussian distribution

$$P(a_n) = P(\xi_n)P(\zeta_n), \quad (18)$$

$$P(\xi_n) = \frac{1}{(2\pi\langle \xi_n^2 \rangle)^{1/2}} \exp \left( -\frac{\xi_n^2}{2\langle \xi_n^2 \rangle} \right), \quad (19)$$

$$\langle \xi_n^2 \rangle = \frac{1}{2M} \sum_{m=1}^M |a_n^m|^2 = \frac{1}{2} \langle |a_n|^2 \rangle. \quad (20)$$

Obviously, Eqs. (18)–(20) are equivalent to Eq. (15).  $\langle |a_n|^2 \rangle$  is a function of  $n$  and  $T$ ; so far, we have not specified the value of  $T$ . But in the limit  $T \gg \tau_c$  we have always found numerically that it becomes a function of single parameter  $\omega$  and independent of  $T$ .

Due to a general nature of our arguments, they can be directly applied to a continuous time, namely chaotic motion described by differential equations.

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<sup>1</sup>L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Addison-Wesley, Reading, MA, 1964).

<sup>2</sup>A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983).

<sup>3</sup>E. N. Lorenz, Ann. N.Y. Acad. Sci. **357**, 282 (1980); J. Stat. Phys. **36**, 1 (1984).

<sup>4</sup>L. Bromberg and A. B. Rechester, Bull. Am. Phys. Soc. **29**, 1261 (1984).

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